Synopsis

Title : Generalized continuity, generalized derivative and its application to integration theory.

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There are many generalization of the derivative, such as Peano Derivative, Approximate derivative etc. The main properties of the ordinary derivative also hold for these generalizations. Like Derivative there are many generalization of continuity such as Absolute continuity, Approximate continuity etc. The main properties of ordinary continuity also hold for the above generalizations. We have introduced a new kind of regularity condition for a real function and since it looks like Laplace transform we give the name Laplace continuity. Every ordinary derivative has this property. It may noted that Laplace derivative which is a generalization of Peano derivative is studied in [3] and [2].

We have defined the Laplace Continuity precisely as follows if f is special Denjoy integrable in a neighborhood of x. Then f is said to be Laplace continuous at x if for a fixed $\delta > 0$

$$\lim_{s\to\infty} s\int_0^\delta e^{-st}[f(x+t)-f(x)]dt \text{ and } \lim_{s\to\infty} s\int_0^\delta e^{-st}[f(x-t)-f(x)]dt$$

exist and are equal to zero. We proved by an example that Laplace Continuity is more general than the Continuity. We have proved some properties of Laplace Continuous function such as Baire, Monotonicity, Darboux, Mean Value, and proved that the space $LC[a,b]$ of all bounded Laplace Continuous function functions on an interval $[a,b]$ is a Banach space with respect to sup norm. We prepared a paper [4] containing these properties which is published in " Real Analysis Exchange".We have proved the main results of [2] under the weaker condition that the function is Laplace Continuous. We shall investigate the properties of the function in a more general function space $LC[a,b]$.

We have introduced the notion of Approximate Generalized Riemann Derivative of order two . It is defined as:

Let l be a non-negative integer and L be a non zero real number.Let

$$A = b_0, b_1, ...b_{2+l}; a_0, a_1, ...a_{2+l}$$

be a system of real numbers such that $b_i \neq b_j$ for $i \neq j$; $\quad i, j = 0, 1, 2, ...2 + l$. Let,

$$\sum_{i=0}^{2+l} a_i = 0,$$

$$\sum_{i=0}^{2+l} a_i b_i^p = \begin{cases} 0 & \text{for } \ p = 1 \\ L & \text{for } \ p = 2 \end{cases}$$

Now let us define,

$$\overline{GRD}_a^2 f = \lim_{h \to 0} ap \sup [\frac{2!}{Lh^2} (\sum_{i=0}^{2+l} a_i f(x + b_i h))]$$

$$\underline{GRD}_a^2 f = \lim_{h \to 0} ap \inf [\frac{2!}{Lh^2} (\sum_{i=0}^{2+l} a_i f(x + b_i h))]$$

and call them the upper and lower approximate generalized Riemann derivative of order 2 at x. If the above two limits exists and are equal, the common value is called approximate generalized Rreimann derivative of order 2 at x and is denoted by $GRD_a^2 f$. We have proved the convexity of f under the condition that f is approximate continuous and the approximate generalized Riemann derivative of order 2 is non-negative with the help of approximate maxima [8]. We have also introduced the notion of approximate generalized symmetric derivative of order two $GRD_a^2 f$ and proved that in the above convexity conditions $\overline{GRD}_a^2 f$ can be replaced by $\overline{GSD}_a^2 f$. A paper containing these properties have published [9].

We have defined the D_k-Integral as,

A function $f : [a, b] \to R$ is said to be D_k integrable on $[a, b]$ if there exists a continuous function $\phi : [a, b] \to R$ such that

i) $\phi \in AC_k G([a, b])$

ii) $\phi_{ap}^{(k-2)}$ exists in $[a, b]$

iii) $\phi_{ap}^{(r)}$ is smooth in (a, b) for $r = 0, 1, ..., k - 2$.

iv) $\phi_{ap}^{(k-1),+}(a)$, $\phi_{ap}^{(k-1),-}(b)$ exists and $\phi_{ap}^{(k)} = f$ a.e in $[a, b]$. Where $\phi_{ap}^{(k-1),+}(a)$ and $\phi_{ap}^{(k-1),-}(b)$ denote right hand approximate derivative of ϕ at a of order $k - 1$ and left hand approximate derivative of ϕ at b of order $k - 1$.

The function ϕ if exists is called k^{th} primitive of f on $[a, b]$. If $F = \phi_{ap}^{(k-1)}$ we call F to be an indefinite D_k integral of f and

$$\phi_{ap}^{(k-1),-}(b) - \phi_{ap}^{(k-1),+}(a) = F(b) - F(a)$$

is called the definite D_k integral of f over $[a, b]$and is denoted by

$$(D_k) \int_a^b f(t)dt.$$

Integration by parts formula for this integral is proved and published [11].

We have defined the Perron type integral using first order Laplace Derivative and proved the integration by parts formula for this integral.This work has also published [10]. We have also defined the Perron type integral using higher order Laplace Derivative and studied properties of this integral which is published in the paper "A Perron Type Integration Using Laplace Derivative" [13]. Similarly we shall try to define integration using the above generalized Riemann derivative and study the suitable properties to apply it in Trigonometric series. There are many applications of of symmetric derivative in Trigonometric series via-integration theory.We shall investigate whether our generalized symmetric derivative is suitable for this application.

Mukhopadhyay and Ray used a special technique to define L_p-derivative in their paper "Relation between L_p-derivates and Peano, approximate Peano and Borel derivates of higher order "[12] and relation between L_p-derivatives and Peano, Approximate Peano and Borel derivatives are studied. We are trying to use the same technique for the symmetric L_p-derivative and studying the relation between symmetric d.l.V.P. derivative and symmetric Borel derivative. We are also trying to use this technique to define L_p Riemann derivative and studying its properties.

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